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Algebraic structure of a generalized coupled dispersionless system

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Abstract
We study a physical model of the $O(3)$-invariant coupled integrable dispersionless equations that describes the dynamic of a focused system within the background of a plane gravitational field. The investigation is carried out both numerically and analytically, and realized beneath some assumptions superseding the structure constant with the structure function implemented in Lie algebra and quasigroup theory, respectively. The energy density and topological structures such as loop soliton are examined.

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1. Introduction

Nonlinear equations play a central role in modern science. In particular, ordinary differential equations (ODEs) and partial differential equations (PDEs) of nonlinear type are very often encountered in the theoretical description of a broad variety of phenomena and processes. Examples are found in various disciplines such as classical mechanics, biology, chemistry and electronics, to name a few.

In addition, the focal point of the study of any nonlinear PDE is the question of its integrability. There exist three approaches to this question namely: Lie analysis, numerical studies and Painlevé analysis. The third analysis requires that the nonlinear PDE is integrable if and only if it possesses the Painlevé property [1].

During the past several years, it has been seen that the study of nonlinear evolution equations has attracted many mathematicians and theoretical physicists due to its considerable applications in various branches of science [2]. The study of nonlinear phenomena has been very interesting and challenging mathematically and physically in recent years. Considerable interest has been paid recently to dispersionless or quasiclassical limits of integrable equations

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and hierarchies [3]. Study of dispersionless hierarchies is of great importance since they arise in the analysis of various problems in physics, mathematics and applied mathematics from the theory of conformal maps on the complex plane. Different methods have been used to study dispersionless equations and hierarchies [3]. In particular, several (1+1)-dimensional equations and systems have been analysed by the quasiclassical version of the inverse scattering transform (IST), including the local Riemann–Hilbert problem approach.

The dispersionless integrable hierarchies can be viewed as quasiclassical limit of the ordinary integrable systems [4]. A typical example is the dispersionless Kadomtsev–Petviashvili (dKP) hierarchies which have played an important role in theoretical and mathematical physics [5]. The Lax formulation of the dKP hierarchy can be constructed by replacing the pseudodifferential Lax operator of KP with the corresponding Laurent series. On the other hand, an analogous construction can be made for the modified KP (mKP) hierarchy and thus leads to the dmKP hierarchy.

The singular point structure analysis leading to the Painlevé (P-)property for ordinary differential equations [6] plays a very useful role in determining the integrability property of nonlinear dynamical system [7, 8]. Weiss *et al* [9] reformulated and generalized the P-test for PDEs [10]. When compared with the uncoupled systems, many coupled systems are not completely analysed because of the complicated and tedious mathematical analysis involved in understanding the nature of their dynamics. However, the P-analysis of the coupled nonlinear Schrödinger (NLS) equation, higher order coupled NLS, nonlinear coupled Klein–Gordon equation [11], inhomogeneous coupled NLS, nonlinear coupled integrable dispersionless equations [12], and so on, have been investigated.

In recent past years, the integrable system of coupled integrable dispersionless equations has been studied by many authors [13, 14]. Some authors [13] have presented and solved the above system by the IST technique. From this method, they solved the Gel'fang–Levitan equations. Kotlyarov [15] proved that these integrable models are gauge equivalent to the sine-Gordon and Pohlmeyer-Lund-Regge models. Again, Konno and Kakuhata have investigated the IST method and obtained the soliton solutions for growing, decaying and stationary solutions. Investigation among the solitary waves and their integrability properties are considered. In another paper, the same authors have solved the system by the IST method and discussed one- and two-soliton solutions. Even though the coupled integrable dispersionless equations are known to be completely integrable, their P-property has been established. The remarkable feature of the P-analysis, particularly for soliton solutions, is that a natural connection exists between Lax pair, Bäcklund transform (BT), Hirota bilinear form and Miura transformation which can be constructed through the expansion of the solutions about the singularity manifold.

The IST scheme for soliton equations is a powerful tool for obtaining N-soliton solutions and an infinite number of conserved quantities. The most famous one is the ZS-AKNS scheme [16, 17]. Many inverse scattering schemes, such as the ZS-AKNS and its varieties, have 2×2 matrix form. There are, however, fewer generalizations of them to 3×3 or higher dimensional matrix forms. It is interesting to hunt for soliton equations with a general $n \times n$ inverse scattering scheme.

Recently, some authors [18] proposed a generalized coupled dispersionless system given by

$$\partial_{tx}^2 S + [\partial_x S, [S, G]] = 0, \tag{1}$$

where the matrix $S = S(t, x)$ and the constant matrix G are elements of an arbitrary non-Abelian Lie algebra. This equation has the $n \times n$ ZS-AKNS-type IST scheme and nonlinearity comes from the non-Abelian character. Equation (1) is a generalization of the coupled integrable, dispersionless equation

$$\partial_{tx}^2 q + \tfrac{1}{2}\partial_x(rs) = 0, \qquad \partial_{tx}^2 r - r\partial_x q = 0, \qquad \partial_{tx}^2 s - s\partial_x q = 0, \tag{2}$$

based on a group-theoretical point of view. For $SU(1,1) \sim O(2,1) \sim SL(2,R)$, equation (1) reproduces (2). For $SU(2) \sim O(3)$, we can obtain

$$\partial_{tx}^2 q + \tfrac{1}{2}\partial_x(rr^\star) = 0, \qquad \partial_{tx}^2 r - r\partial_x q = 0, \qquad \partial_{tx}^2 r^\star - r^\star\partial_x q = 0, \tag{3}$$

which can be equivalent to the Pohlmeyer-Lund-Regge system. Note that the star refers to conjugation. Equations (2) and (3) have been solved by the IST scheme under the appropriate boundary conditions and shown to be integrable. They have the important conserved quantities

$$\partial_x q^2 + \partial_x r\partial_x s = q_0^2, \qquad \partial_x q^2 + \partial_x r\partial_x r^\star = q_0^2, \tag{4}$$

which are obtained from the IST scheme. Here $q_0 = \partial_x q(\pm\infty)$ is constant.

Furthermore, some connection of theoretical physics with a nonassociativity algebra and differential geometry has been established. This connection has helped solving many problems in physics. Indeed, during the last 30 years quite remarkable relations between the nonassociative algebra and differential geometry have been discovered. Such exotic structures of algebra as quasigroups and loops were obtained from purely geometric structures such as affinely connected spaces. The notion of odule was introduced as a fundamental algebraic invariant of differential geometry. For any space with an affine connection, loopuscular, odular and geoodular structures (partial smooth algebra of a special kind) were introduced and studied. There are now three main approaches in theoretical physics exploring the notion of a nonassociativity system:

- octonionic approach;
- Lie-admissible approach;
- quasigroup approach.

We will focus only on the last. It is essentially based on new nonassociative algebraic methods in differential geometry where the local properties of some global continuous structures such as quasigroups, loops, etc., have been studied. The recent development has demonstrated that also various nonassociativity systems, such as quasigroups, loops, odules, etc., play important roles in geometry and also in physical applications.

In physics, the main stimulation of the quasigroup approach is provided by modern gauge theories, quantum gravity and some attempts of extension or generalization of the classical method of symmetry and invariance. Here we shall make some brief comments about these directions.

Nowadays, gauge theories based on continuous (Lie) groups have become an essential part of modern theoretical physics providing the unified treatment of fundamental forces of nature through the localization (gauging) of the global group symmetry. Recently, this approach has been generalized for quasigroups by some authors [19].

Recently, a purely algebraic formulation of differential geometry, nonlinear geometric algebra, has been elaborated by some authors [20, 21]. In this approach, nonassociativity appears as an algebraic equivalent of the geometrical notion of curvature. This geometric algebra provides a new algebraic approach to the theory of gravity, where the spacetime is considered as an algebraic system with a geodesic multiplication of points in a certain way. The curvature of spacetime is then expressed by the nonassociativity of this multiplication. There is hope for arranging another attack upon the most actual problem of gravity—the quantization of gravity.

Our aim in this paper is to reconsider some differential system of equations modelling the behaviour of a charged particle within a magnetic field, and to undergo some computations both analytically and numerically from a curved-space perspective. In this view, our work is planned as follows. In section 2, we review the smooth loop theory briefly, then a generalization of the system under our interest is given in section 3. Next, some application is done in section 4 dealing with the background of a weak plane gravitational wave. Lastly, we end our work with some concluding remarks.

2. Smooth loop

2.1. Basic smooth structures

2.1.1. Smooth local loops (loopusculas)

2.1.1.1. Definition. Let $\varphi : M \times \cdots \times M$ be a partial m-ary operation on a C^k-smooth manifold M such that $\varphi(a_1, \ldots, a_m) = b$ (i.e. φ is defined on $a_1, \ldots, a_m$) [20] then there exist open sub-manifolds $U_1, \ldots, U_m$ containing $a_1, \ldots, a_m$ respectively, φ being defined on $U_1 \times \cdots \times U_m$ and the restriction $\varphi|_{U_1\times\cdots\times U_m} : U_1 \times \cdots \times U_m \longrightarrow M$ is a C^r-smooth mapping $(r \leqslant k)$. Then φ is said to be a C^r-smooth partial m-ary operation on C^k-smooth manifold.

If φ is defined everywhere on M then we say that φ is a C^r-smooth global m-ary operation.

A C^k-smooth manifold M equipped with a family of C^r-smooth partial (global) operations $(r \leqslant k)$ and a family of constants (fixed elements) is called a $C^{r,k}$-smooth partial (global) algebra (C^r-smooth partial algebra if $r = k$).

2.1.1.2. Definition. Let $\langle M, \cdot, \varepsilon\rangle$ be a partial magma (groupoid) with a binary operation $(x, y) \longmapsto x \cdot y$ and the neutral element ε, M being a C^k-smooth manifold and the operation of multiplication (at least C^1-smooth) being defined in some neighbourhood $U \ni \varepsilon$ [20]. As is known, the above operation is locally left and right invertible i.e. if $x \cdot y = L_x y = R_y x$ then in some neighbourhood of the neutral element ε there exist L_x^{-1} and R_x^{-1}. This will allow us to introduce left and right division

$$a \setminus x = L_a^{-1} x, \qquad x/b = R_b^{-1} x, \tag{5}$$

with properties

$$\begin{aligned} a \cdot (a \setminus x) &= x, \qquad (x/b) \cdot b = x, \\ a \setminus (a \cdot x) &= x, \qquad (x \cdot b)/b = x. \end{aligned} \tag{6}$$

Thus we have, indeed, a partial loop on M.

2.1.2. Canonical odules and odular structures

2.1.2.1. Definition. Let $\langle Q, \cdot, \setminus, \varepsilon\rangle$ be a partial left loop with two-sided neutral ε defined on C^k-smooth manifold Q (dim$Q = n$) [20]. We say that $A_1, \ldots, A_n$ are the left basic fundamental vector field of Q if

$$\begin{aligned} [A_\alpha(x)]^\beta &= A_\alpha^\beta(x) \\ &= \left[\frac{\partial (x \cdot y)^\beta}{\partial y^\alpha}\right]_{y=\varepsilon}. \end{aligned} \tag{7}$$

Any $A = \zeta^\alpha A_\alpha(\zeta^1, \ldots, \zeta^n \in \mathbb{R})$ is called a left fundamental vector field in the case.

2.1.2.2. Definition. Let Q be a C^k-manifold. A partial left loop $\langle Q, \cdot, \backslash, \varepsilon\rangle$ is called a left$(,k)$-canonical ($p \geqslant 1$) if $L_x : y \longmapsto x \cdot y$ is C^1-smooth near ε and its left fundamental vector fields are C^p-smooth near ε. In the case $p = 1$, we say left canonical instead of left (p, k)-canonical. Analogously [20] one can define a right (p, k)-canonical loop, replacing the left basic fundamental vector field by right ones, $a_1, \ldots, a_n$ such that

$$[a_\alpha(y)]^\beta = a_\alpha^\beta(y) = \left[\frac{\partial(x\cdot y)^\beta}{\partial x^\alpha}\right]_{x=\varepsilon}. \tag{8}$$

2.2. Infinitesimal theory of smooth loop: general theory

2.2.1. Determination of φ^α and $\widetilde{l}^\alpha_\beta$. Let $\langle Q, \cdot, \varepsilon\rangle$ [20] be a smooth partial loop with the neutral ε. Let us introduce the following notations:

$$L_a b = a\cdot b, \qquad l(a,b) = L^{-1}_{a\cdot y} o L_a o L_b, \qquad \widetilde{l}(a,b) = [l(a,b)]_{\star,\varepsilon},$$
$$A^\alpha_\beta(a) = [(L_a)_{\star,\varepsilon}]^\alpha_\beta, \qquad B^\lambda_\mu(a) = \left[(L_a)^{-1}_{\star,\varepsilon}\right]^\lambda_\mu, \tag{9}$$

where $\alpha, \beta, \lambda, \mu = 1, \ldots, n = \dim M$. Differentiating the relation $(a\cdot b)\cdot l(a,b)c = a\cdot(b\cdot c)$ by c at $c = \varepsilon$, we have

$$\frac{\partial(a\cdot b)^\lambda}{\partial b^\mu} = A^\lambda_\gamma(a\cdot b)\widetilde{l}^\gamma_\sigma(a,b)B^\sigma_\mu(b). \tag{10}$$

Let us introduce the vector fields A_γ and covector fields B^ν by formulae

$$(A_\gamma(a))^\lambda = A^\lambda_\gamma(a), \qquad (B^\nu(a))_\beta = B^\mu_\beta(a), \tag{11}$$

$$[A_\alpha, A_\beta](a) = C^\gamma_{\alpha\beta}(a)A_\gamma(a), \qquad C^\gamma_{\alpha\beta} = -C^\gamma_{\beta\alpha}, \tag{12}$$

where $C^\gamma_{\alpha\beta}$ are defined as the structure functions of point. Following equation (12), we get the relation

$$A^\gamma_\alpha(a)\partial_\gamma A^\lambda_\beta(a) - A^\gamma_\beta(a)\partial_\gamma A^\lambda_\alpha(a) = C^\gamma_{\alpha\beta}(a)A^\lambda_\gamma(a), \tag{13}$$

which is equivalent to

$$B^\nu_\lambda(a)\left[A^\gamma_\alpha(a)\partial_\gamma A^\lambda_\beta(a) - A^\gamma_\beta(a)\partial_\gamma A^\lambda_\alpha(a)\right] = C^\gamma_{\alpha\beta}(a), \tag{14}$$

or briefly

$$B^\nu([A_\alpha, A_\beta]) = C^\gamma_{\alpha\beta}. \tag{15}$$

Using Jacobi's identity

$$[A_\alpha,[A_\beta,A_\gamma]] + [A_\gamma,[A_\alpha,A_\beta]] + [A_\beta,[A_\gamma,A_\alpha]] = 0, \tag{16}$$

we have

$$A^\sigma_{<\alpha}(a)\partial_{|\sigma|}C^\mu_{\beta\gamma>}(a) + C^\mu_{<\alpha|\nu|}(a)C^\nu_{\beta\gamma>}(a) = 0. \tag{17}$$

(Here $\langle\alpha, \beta, \gamma\rangle$ means the cyclic sum over α, β, γ)

Assuming that

$$\frac{\partial^2(a\cdot b)^\alpha}{\partial b^\nu \partial b^\mu} = \frac{\partial^2(a\cdot b)^\alpha}{\partial b^\mu \partial b^\nu}, \tag{18}$$

we obtain

$$A^\sigma_\nu(b)\frac{\partial\widetilde{l}^\gamma_\mu(a,b)}{\partial b^\sigma} - A^\sigma_\mu(a)\frac{\partial\widetilde{l}^\gamma_\nu(a,b)}{\partial b^\sigma} = C^\sigma_{\nu\mu}(b)\widetilde{l}^\gamma_\sigma(a,b) - C^\gamma_{\tau\lambda}(a\cdot b)\widetilde{l}^\tau_\nu(a,b)\widetilde{l}^\lambda_\mu(a,b). \tag{19}$$

In particular at $b = \varepsilon$, we have

$$C^{\gamma}_{\nu\mu}(a) = C^{\gamma}_{\nu\mu}(\varepsilon) + \left[\frac{\partial \widetilde{l}^{\gamma}_{\nu}(a,b)}{\partial b^{\mu}} - \frac{\partial \widetilde{l}^{\gamma}_{\mu}(a,b)}{\partial b^{\nu}}\right]_{b=\varepsilon}. \tag{20}$$

This result is of good promise since it gives an assessment of the value of the structure function on a given manifold.

2.2.2. Proposition. Let $\langle Q, \cdot, \varepsilon\rangle$ be a smooth partial loop. Then $\varphi^{\alpha} = (a \cdot b)^{\alpha}$ and $\widetilde{l}^{\gamma}_{\mu} = \widetilde{l}^{\gamma}_{\mu}(a, b)$ are the solutions of the system of differential equations [20]

$$\begin{aligned} &\frac{\partial \varphi^{\alpha}}{\partial b^{\mu}} = A^{\alpha}_{\gamma}(\varphi)\widetilde{l}^{\gamma}_{\sigma} B^{\sigma}_{\mu}(b), \\ &A^{\sigma}_{\nu}(b)\frac{\partial \widetilde{l}^{\alpha}_{\mu}}{\partial b^{\sigma}} - A^{\sigma}_{\mu}(b)\frac{\partial \widetilde{l}^{\alpha}_{\nu}}{\partial b^{\sigma}} = C^{\sigma}_{\nu\mu}(b)\widetilde{l}^{\alpha}_{\sigma} - C^{\alpha}_{\tau\lambda}(\varphi)\widetilde{l}^{\tau}_{\nu}\widetilde{l}^{\lambda}_{\mu}, \\ &\varphi^{\alpha}|_{\varepsilon} = a^{\alpha}, \qquad \widetilde{l}^{\nu}_{\tau}\big|_{\varepsilon} = \delta^{\nu}_{\tau}. \end{aligned} \tag{21}$$

The functions $A^{\alpha}_{\gamma}(b)$ are supposed to be given and satisfy the conditions $A^{\alpha}_{\gamma}(\varepsilon) = \delta^{\alpha}_{\gamma}$.

Remark 1. There exist some link between the structure function and the curvature tensor of a manifold [20]. Let us introduce the following infinitesimal right and left translation matrices

$$\begin{aligned} (x \cdot y)^{\mu} &= y^{\mu} + L^{\mu}_{\nu}(y)x^{\nu} + \cdots, \\ (x \cdot y)^{\mu} &= x^{\mu} + R^{\mu}_{\nu}(x)y^{\nu} + \cdots, \end{aligned} \tag{22}$$

with $L^{\mu}_{\nu}(y) = \frac{\partial (x\cdot y)^{\mu}}{\partial x^{\nu}}\big|_{x=e}$ and $R^{\mu}_{\nu}(x) = \frac{\partial (x\cdot y)^{\mu}}{\partial y^{\nu}}\big|_{y=e}$. Matrices $R^{\mu}_{\nu}(x)$ and $L^{\mu}_{\nu}(y)$ can be used to introduce a local frame field

$$R_{\nu}(x) = R^{\mu}_{\nu}(x)\partial_{\mu}, \qquad L_{\nu}(y) = L^{\mu}_{\nu}(y)\partial_{\mu}. \tag{23}$$

It is well known that for two vector fields, their commutator is a vector field. We know that $L_{\nu}(x)$ and $R_{\nu}(x)$ are frame fields, so it is quite natural to define the structure functions $\lambda^{\gamma}_{\mu\nu}(x)$ and $C^{\gamma}_{\mu\nu}(x)$ by

$$[L_{\mu}(x), L_{\nu}(x)] = -\lambda^{\gamma}_{\mu\nu}(x)L_{\gamma}(x), \tag{24}$$

$$[R_{\mu}(x), R_{\nu}(x)] = -C^{\gamma}_{\mu\nu}(x)R_{\gamma}(x). \tag{25}$$

In general, the structure functions $\lambda^{\gamma}_{\mu\nu}(x)$ and $C^{\gamma}_{\mu\nu}(x)$ do not coincide. Those functions have expansions

$$\lambda^{\gamma}_{\mu\nu}(x) = -R^{\gamma}_{[\mu\nu]\delta}(e)x^{\delta} + \cdots, \tag{26}$$

$$C^{\gamma}_{\mu\nu}(x) = -R^{\gamma}_{\delta[\mu\nu]}(e)x^{\delta} + \cdots. \tag{27}$$

The commutator of two frame fields can also be calculated

$$[L_{\mu}(x), R_{\nu}(x)] = -\tfrac{1}{2}R^{\gamma}_{\mu\delta\nu}(e)y^{\delta}\partial_{\gamma} + \cdots. \tag{28}$$

Here $R^{\gamma}_{\mu\delta\nu}(e)$ denotes components of the curvature tensor at the unit element e.

3. The extent generalized coupled dispersionless system

Let G be a simple extent Lie group, $\dim G = n$, and $\mathcal{G}$ be its extent Lie algebra. The generators $T^a(x)$ of $\mathcal{G}$ (where $x = (x^\mu, \mu = 0, 1, \ldots)$) satisfy the commutation relation with the structure function [21] $C_c^{ab}(x)$ such that

$$[T^a(x), T^b(x)] = C_c^{ab}(x) T^c(x), \tag{29}$$

and the Cartan metric η^{ab} defined by

$$\eta^{ab} = \mathrm{Tr}(T^a T^b). \tag{30}$$

Without any loss of generality, we can take η^{ab} as diagonal and $C_c^{ab}(x)$ as totally antisymmetric; with T^a's we define S by

$$\begin{aligned} S &= \phi_a(x) T^a(x) \\ &= \eta_{ab} \phi^a(x) T^b(x), \end{aligned} \tag{31}$$

where $\phi^a = \phi^a(x, t)$ is a vector field with components $(\phi^1, \phi^2, \ldots, \phi^N)$ and η_{ab} is the inverse matrix of η^{ab}. We also define a matrix function $\mathcal{M}$ as

$$\mathcal{M} = k_a(x) T^a(x), \tag{32}$$

with an analytical vector $k^a = (k^1, k^2, \ldots, k^N)$. These quantities are rotated by the global gauge transformation

$$S' = \Omega^{-1} S \Omega, \qquad \mathcal{M}' = \Omega^{-1} G \Omega, \tag{33}$$

where $\Omega \in G$.

Let us write the action of the generalized coupled dispersionless system [22] as

$$I = \int \mathrm{d}t\, \mathrm{d}x \mathcal{L}(S, \partial_\mu S, \partial_0 S), \tag{34}$$

where $\mathcal{L}$ is the Lagrangian density defined by

$$\mathcal{L} = \epsilon_0^\mu \, \mathrm{Tr}\big(\tfrac{1}{2} \partial_\mu S \partial_0 S - \tfrac{1}{3} \mathcal{M}[S, [\partial_\mu S, S]]\big). \tag{35}$$

Note that the ϵ-character represents the Levi-Civita symbol. The Lagrangian density is manifestly invariant under the global gauge transformation (33). By using the Euler–Lagrange equation, we obtain

$$\epsilon_0^\mu \big(\partial_{0\mu}^2 S - [[S, \mathcal{M}], \partial_\mu S]\big) = 0. \tag{36}$$

From equation (36), we show that $\mathrm{Tr}(\partial_\mu S)^n$ is conserved for integer n ($n \geqslant 2$) and is also invariant under gauge transformation. Equation (36) is known as the generalized coupled dispersionless system [22]. Positing $\partial_\mu T^a = B_{\mu c}^a T^c$ and $\partial_\mu T^a = D_{\mu c}^a T^c$ we get

$$\begin{aligned} &[S, \mathcal{M}] = \phi_a k_b C_c^{ab} T^c, \\ &[[S, \mathcal{M}], \partial_\mu S] = \big\{(\partial_\mu \phi_d) \phi_e k_b C_c^{eb} C_a^{dc} + \phi_d \phi_e k_b C_c^{eb} B_{\mu f}^d C_a^{fc}\big\} T^a \\ &\partial_\mu \partial_0 S = \big\{\partial_\mu \partial_0 \phi_a + D_{0a}^b \partial_\mu \phi_b + (\partial_0 \phi_d) B_{\mu a}^d + \phi_d \partial_0 B_{\mu a}^d + (\partial_0 \phi_d) B_{\mu b}^d D_{0a}^d\big\} T^a, \end{aligned} \tag{37}$$

and hence

$$\begin{aligned} \epsilon_0^\mu \big\{&\partial_\mu \partial_0 \phi_a + D_{0a}^b \partial_\mu \phi_b + (\partial_0 \phi_d) B_{\mu a}^d + \phi_d \partial_0 B_{\mu a}^d + (\partial_0 \phi_d) B_{\mu b}^d D_{0a}^d \\ &+ (\partial_\mu \phi_d) \phi_e k_b C_c^{eb} C_a^{dc} + \phi_d \phi_e k_b C_c^{eb} B_{\mu f}^d C_a^{fc}\big\} = 0. \end{aligned} \tag{38}$$

We consider the case of a single j-coordinate and set

$$C_c^{ab} = \imath f_c^{ab} q(x^j), \qquad B_{\mu b}^a = \epsilon_{\mu b}^a u(x^j, t),$$
$$D_{0a}^b = \epsilon_{0a}^b v(x^j, t), \qquad \tau = x^j - \xi x^0, \quad \sigma = x^j + \xi x^0, \tag{39}$$

where u, v, q are arbitrary functions and ξ a constant. By a suitable redefinition of x^0 leading to $\xi = 1$, it comes

$$\partial_\sigma^2 \phi - \partial_\tau^2 \phi - q^2(\partial_\sigma \phi + \partial_\tau \phi) \times (\phi \times k) = (u + v)\partial_\sigma \Phi + (v - u)\partial_\tau \Phi$$
$$+ (\partial_\sigma u - \partial_\tau v)\Phi + uv\varphi - q^2 u\, [\Phi \times (\phi \times k)]\,, \tag{40}$$

where $\Phi_a = \epsilon_a^b \phi_b$ and $\varphi_a = \epsilon_a^b(\partial_\sigma \Phi_b - \partial_\tau \Phi_b)$.

We emphasize here that the case $u = 0$, $v = 0$, $q = 1$ was already obtained in [22]; using Hirota's method with restrictions on second-order expansions, loop solitons were derived as the essence of the response of elastica matter to focusing excitations. Let us give briefly the basic start points used. With the following settings

$$\phi_1 = \frac{W}{F}, \qquad \phi_2 = \frac{H}{F},$$
$$\phi_3 = \sigma + 2(\partial_\tau - \partial_\sigma) \ln F, \tag{41}$$

one derives the following bilinear equations:

$$\left(D_\tau^2 - D_\sigma^2 + 1\right) F \cdot W = 0, \qquad \left(D_\tau^2 - D_\sigma^2 + 1\right) F \cdot H = 0,$$
$$(D_\tau - D_\sigma)^2 F \cdot F - \tfrac{1}{2}(W^2 + H^2) = 0, \tag{42}$$

where D_s denotes Hirota's derivatives. This system is equivalent to

$$W F_{\tau\tau} + F W_{\tau\tau} - W F_{\sigma\sigma} - F W_{\sigma\sigma} - 2F_\tau W_\tau + 2F_\sigma W_\sigma + FW = 0,$$
$$H F_{\tau\tau} + F H_{\tau\tau} - H F_{\sigma\sigma} - F H_{\sigma\sigma} - 2F_\tau H_\tau + 2F_\sigma H_\sigma + FH = 0, \tag{43}$$
$$2\left(F F_{\tau\tau} + F F_{\sigma\sigma} - F_\sigma^2 - F_\tau^2\right) - \tfrac{1}{2} W^2 - 4(F F_{\sigma\tau} - F_\tau F_\sigma) = 0.$$

Then we expand F, W, H into convenient power series to derive loop structures [22]. Instead of using the previous analytic technique which does not work easily with complex-related systems, we can try some numerical method suitable with some boundary conditions. We can look for solitary waves by setting $s = \sigma - \Sigma\tau$. Therefore,

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} - \frac{q^2}{1+\Sigma}\frac{\mathrm{d}\phi}{\mathrm{d}s} \times (\phi \times k) = \left(\frac{u}{1-\Sigma} + \frac{v}{1+\Sigma}\right)\frac{\mathrm{d}\phi}{\mathrm{d}s}$$
$$+ \frac{\Phi}{1-\Sigma}\frac{\mathrm{d}u}{\mathrm{d}s} + \frac{uv\varphi}{1-\Sigma^2} - \frac{q^2 u}{1-\Sigma^2}[\Phi \times (\phi \times k)], \tag{44}$$

with Σ being a phase velocity.

Now, according to the usual procedure, the Hamiltonian $\mathbf{H}$ is given by

$$\mathbf{H} = \int \mathrm{d}x\, \mathcal{H}, \tag{45}$$

with $\mathcal{H}$ being the Hamiltonian density defined by

$$\mathcal{H} = \partial_0 \phi \cdot \pi - \mathcal{L}$$
$$= \mathrm{Tr}\left(\tfrac{1}{3}\mathcal{M}[S, [\partial_\mu S, S]]\right), \tag{46}$$

π is the canonical conjugate momentum to ϕ. This gives

$$\mathcal{H} = -\frac{q^2}{3} k \cdot [\phi \times (\partial_j \phi \times \phi + u\Phi \times \phi)] - uv\Phi_c\Phi^c - (u+v)\Phi^b\phi_b. \tag{47}$$

It is noteworthy to emphasize here that the symbol $(\cdot)$ denotes the inner product defined by $\varphi \cdot \psi = \varphi^a \psi_a$ and $(\times)$ the exterior product defined by $(\varphi \times \psi)_c = \epsilon_c^{ab} \varphi_a \psi_b$.

Let us consider $\phi = (\phi_1, \phi_2, \phi_3)$ and $k = (0, 0, 1)$; therefore

$$\Phi = (\phi_3 - \phi_2, \phi_1 - \phi_3, \phi_2 - \phi_3), \tag{48}$$

$$\Phi \times (\phi \times k) = \left(\phi_1\phi_2 - \phi_1^2, \phi_2^2 - \phi_1\phi_2, \phi_2\phi_3 - \phi_1\phi_3\right), \tag{49}$$

$$\varphi = \left(\frac{\mathrm{d}\phi_3}{\mathrm{d}s^2} - 2\frac{\mathrm{d}\phi_1}{\mathrm{d}s} + \frac{\mathrm{d}\phi_2}{\mathrm{d}s}, \frac{\mathrm{d}\phi_1}{\mathrm{d}s^2} - 2\frac{\mathrm{d}\phi_2}{\mathrm{d}s} + \frac{\mathrm{d}\phi_3}{\mathrm{d}s}, \frac{\mathrm{d}\phi_2}{\mathrm{d}s^2} - 2\frac{\mathrm{d}\phi_3}{\mathrm{d}s} + \frac{\mathrm{d}\phi_1}{\mathrm{d}s}\right). \tag{50}$$

Besides, by setting

$$\begin{aligned}
F(a,b) &= \left(\frac{u}{1-\Sigma} + \frac{v-2uv}{1+\Sigma}\right)\frac{\mathrm{d}a}{\mathrm{d}s} + \frac{uv}{1+\Sigma}\frac{\mathrm{d}b}{\mathrm{d}s} - \frac{q^2 u}{1-\Sigma^2}(ab - a^2),\\
G(a,b) &= \frac{uv - q^2 a}{1+\Sigma}\frac{\mathrm{d}a}{\mathrm{d}s} + \frac{uv - q^2 b}{1+\Sigma}\frac{\mathrm{d}b}{\mathrm{d}s} + \frac{b-a}{1-\Sigma}\frac{\mathrm{d}u}{\mathrm{d}s},\\
H_1(a,b,c) &= \frac{uv + q^2 a}{1+\Sigma}\frac{\mathrm{d}c}{\mathrm{d}s} + \frac{c-b}{1-\Sigma}\frac{\mathrm{d}u}{\mathrm{d}s},\\
H_2(a,b,c) &= \frac{uv + q^2 b}{1+\Sigma}\frac{\mathrm{d}c}{\mathrm{d}s} + \frac{a-c}{1-\Sigma}\frac{\mathrm{d}u}{\mathrm{d}s},\\
I(a,b,c) &= \left(\frac{u}{1-\Sigma} + \frac{v-2uv}{1+\Sigma}\right)\frac{\mathrm{d}c}{\mathrm{d}s} - \frac{q^2 u}{1-\Sigma^2}(bc - ac),
\end{aligned} \tag{51}$$

we write

$$\begin{aligned}
\frac{\mathrm{d}^2\phi_1}{\mathrm{d}s^2} &= F(\phi_1,\phi_2) + H_1(\phi_1,\phi_2,\phi_3),\\
\frac{\mathrm{d}^2\phi_2}{\mathrm{d}s^2} &= F(\phi_2,\phi_1) + H_2(\phi_1,\phi_2,\phi_3),\\
\frac{\mathrm{d}^2\phi_3}{\mathrm{d}s^2} &= G(\phi_1,\phi_2) + I(\phi_1,\phi_2,\phi_3).
\end{aligned} \tag{52}$$

Also, using the inner and exterior products defined above, we obtain

$$\begin{aligned}
\mathcal{H} = -\frac{q^2}{3}&\left\{|\phi|^2\left[(1-\Sigma)\frac{\mathrm{d}\phi_3}{\mathrm{d}s} + u(\phi_2 - \phi_1)\right] - \phi_3(1-\Sigma)\frac{\mathrm{d}|\phi|^2}{\mathrm{d}s}\right\}\\
&- 2uv\left(|\phi|^2 + \phi_3^2 - \phi_1\phi_2 - \phi_1\phi_3 - \phi_2\phi_3\right),
\end{aligned} \tag{53}$$

which is the Hamiltonian density of the system. Note here that $|\phi|^2 = \phi_1^2 + \phi_2^2$.

4. Some illustration: in the background of a weak plane gravitational wave

Now we give some illustrative values of u, v, q derived from the quantum kinematics survey of some test particle in the background of a weak plane gravitational wave. The metric tensor of the space-time of a weak plane gravitational wave can be given as perturbations around the Minkowski metric [23].

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1). \tag{54}$$

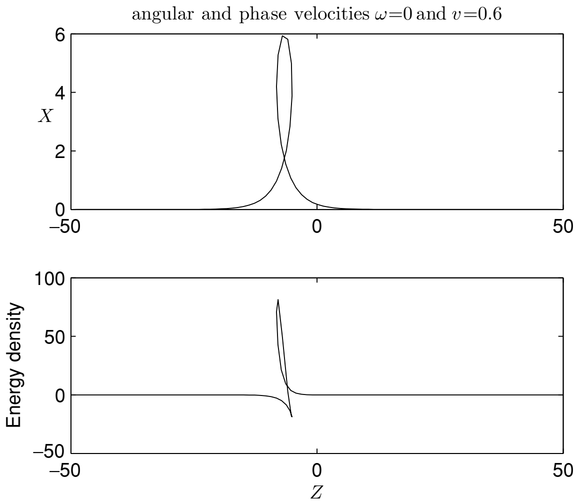

Figure 1. One-loop soliton and energy density in nonperturbed medium.

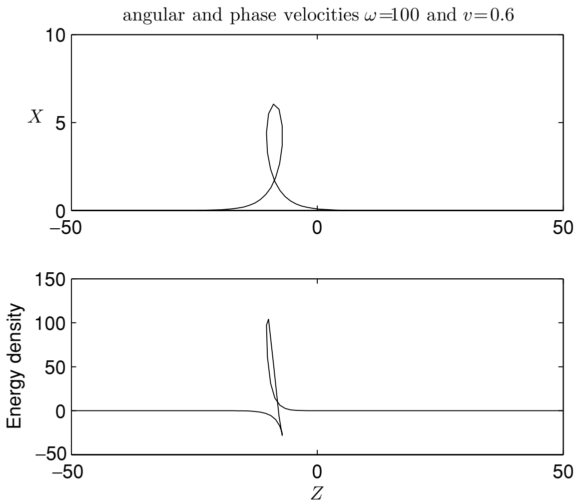

Figure 2. One-loop soliton and energy density in perturbed medium.

In the case of a polarized weak plane gravitational wave moving in the direction of x^j the only non-zero component of $h_{\mu\nu}$ in the TT-gauge are, for example,

$$h_{22} = -h_{33} = A\cos\omega(x^0 - x^1), \tag{55}$$

for $j = 1$. Here $A = \text{const}$, $A \ll 1$, is the wave amplitude. We posit that u, v and q are depending linearly on scalar curvature. Therefore we set

$$q(s) = 1 + \delta A^2 \sin^2\omega s, \qquad u(s) = \eta A^2 \sin^2\omega s, \qquad v(s) = \nu A^2 \sin^2\omega s, \tag{56}$$

where δ, η, ν are some arbitrary constants. Some curves are then derived.

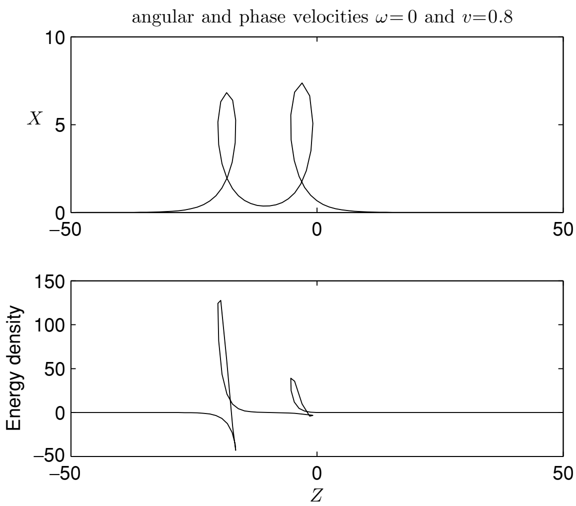

Figure 3. Two-loop soliton and energy density in nonperturbed medium.

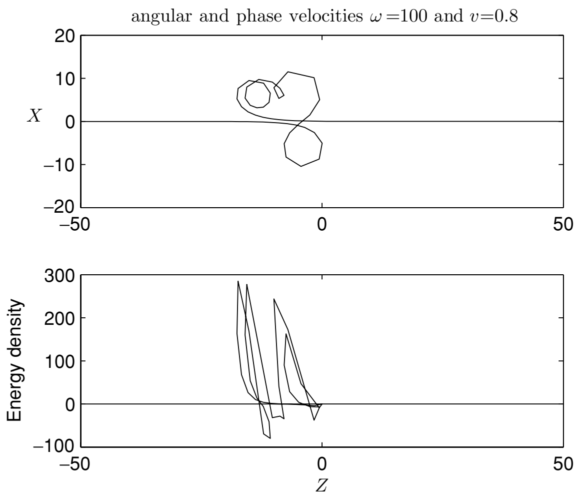

Figure 4. Two-loop soliton and energy density in perturbed medium.

We try some numerical method to plot some curves. As previously done with the loop soliton above in [22], we do consider the same boundary conditions . We will consider two major cases: the first one refers to the absence of perturbation that is $\omega = 0$. In this case, we try a set of plots which will be compared to those obtained analytically from Hirota's method; the second case deals with the influence of the perturbation, that is $\omega \neq 0$. Here we compare some curves we derived with the previous ones related to $\omega = 0$. This may help us assess the genuine role of the angular frequency.

By the way, considering figures 1, 3, 5 and 7 which present plots of $\phi_1 \equiv X$ versus $\phi_3 \equiv Z$ and energy density versus Z, we see one-, two- and three-loop-shaped self-confined structures. This is obtained for angular frequency $\omega = 0$. For the one-loop, we chose a phase

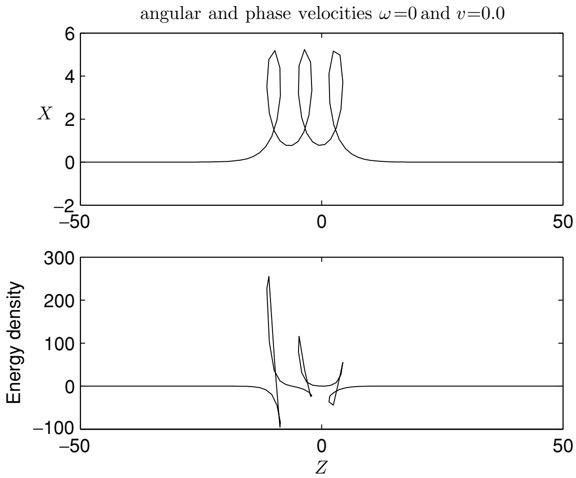

Figure 5. Three-loop soliton and energy density in nonperturbed medium.

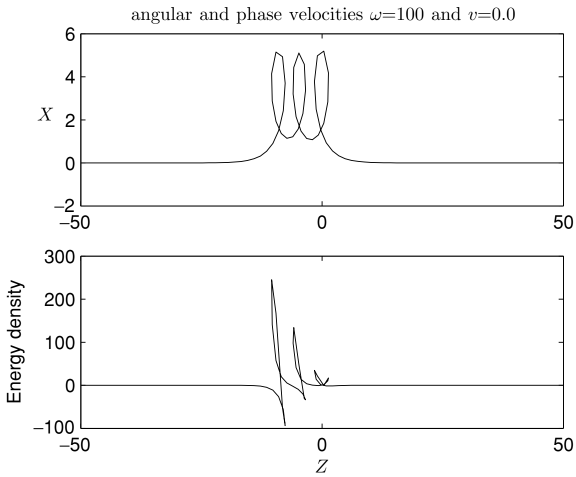

Figure 6. Three-loop Soliton and energy density in perturbed medium.

velocity of $v = 0.6$, $v = 0.8$ for the two-loop, $v = 0.0$ and $v = 0.2$ for the two other. These figures present some fitness between the two methods used so far. The two- and three-loop may be interpreted as the interaction between the two and three single one-loop structure identified previously. With the plot of the energy density, we can get some assessment of the relative stability of the two and three single one-loop solitons interacting. It is worth pointing that for other suitable choices of the phase velocities, one may obtain some *N*-loop structures of interactions. Furthermore, we have found that we may only obtain some loop structures for absolute velocities less than one. This fits well with the above analytical method from Hirota'scheme perspective.

Now, we consider the second case where we choose $\omega = 100$. We then get figures 2, 4, 6 and 8. Comparing these plots to the previous ones of the case $\omega = 0$, there are some

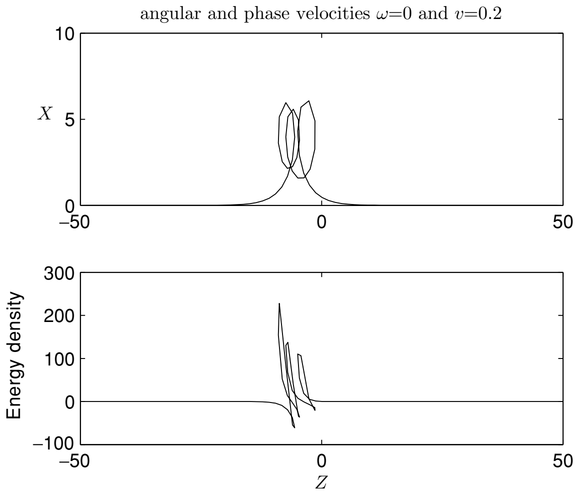

Figure 7. Three-loop soliton and energy density in nonperturbed medium.

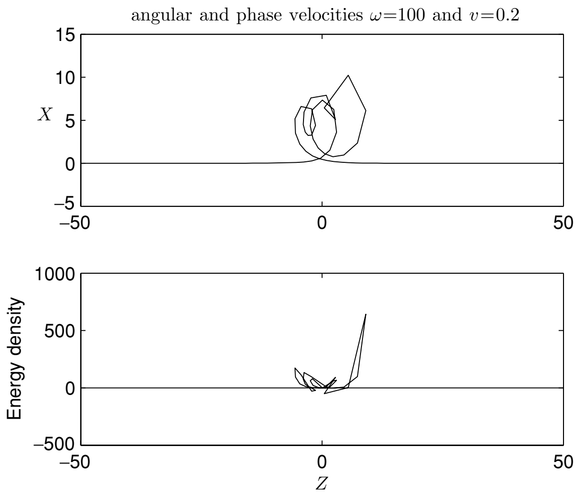

Figure 8. Three-loop soliton and energy density in perturbed medium.

important changes on the shapes. In fact, for $\omega = 100$, we have chosen a real small amplitude of perturbation compared to one. In figure 2, one may see how the small perturbations actions have shifted left-hand side the one-loop soliton leading by this way to a small stretching of the loop. In figure 4, these perturbations have created some distortion on the two-loop case creating two more single loops travelling along the big one. In contrast to the previous case, the effect of the small perturbations on the three-loop soliton as shown in figure 6 is to shift the two single one-loop in the opposite direction so that they would collide with the unmovable centre one. As shown in figure 2, there has been some creation of two more one-loop solitons travelling along two big ones. Then for this velocity of $v = 0.2$, the effect of small perturbations is really important as it is the case of the previous two-loop. It should be noticeable here that the effect of these small perturbations are really seen on the energy density plot which for instance tells us whether there has been some new loop-like soliton.

5. Summary

We have given an extension of the Lagrangian [22] giving rise to coupled dispersionless systems. If we assign the field ϕ_α and the constant k_α to the position vector $\mathbf{r} = (X, Y, Z)$ of the string and the constant external electric field $\mathbf{J}$ respectively, our system behaves like a charged particle moving in an external magnetic field. In contrast to the KdV-type equations in which dispersion effect balances the nonlinearity, our system shows that the nonlinear external force of the dispersionless equations balances the linear elasticity of the string.

With some numerical attempt carried out in this work, we have solved the above nonlinear coupled system within a spacetime universe from a flat perspective. The introduction of a perturbation term has rendered this system not easily tractable at all using Hirota's method. However, this numerical attempt has confirmed the above result derived from this analytical method. As it has been seen, the effect of small perturbation seems to shift the structure under control so that to alter its shape leading to some new structures. This simply means that some gravitational sources such as the spinning double stars, the supernovae explosion may play an essential role on dynamic system modelling, since topological space under interest is locally modified. It is worth noting to see that we could have considered the case of strong fields but the real problem depends on the physical choice of the structure function characterizing this field. This constitutes an open investigation.

Finally, we have also been interested on energy density computations. This has helped us assessing the relative stabilities of some one-loop structures. However, further study in this way should be made so that the relative stabilities of loop-like structures should be assessed.

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